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CONSTRUCTION AND REGULARITY OF MEASURE-VALUED MARKOV BRANCHING PROCESSES

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ABSTRACT

A construction is given for a general class of measure-valued Markov branching processes. The underlying spatial motion process is an arbitrary Borel right Markov process, and state-dependent offspring laws are allowed. It is shown that such processes are Hunt processes in the Ray weak* topology, and have continuous paths if and only if the total mass process is continuous. The entrance spaces of such processes are described explicitly.

1. Introduction

Let $M(E)$ be the space of finite measures on a measurable space $(E, \mathscr{E})$. If $X = (X_t : t \geqq 0)$ is a time homogeneous Markov process with values in $M(E)$, then it is natural to call X a *branching process* provided

$$(1.1)\qquad (X_t : t \geqq 0 \mid X_0 = \mu + \nu) \stackrel{\mathrm{d}}{=} (X'_t + X''_t : t \geqq 0 \mid X'_0 = \mu, X''_0 = \nu)$$

whenever X' and X'' are independent processes with the same transition function as X. Such processes arise as high density limits of certain infinite particle systems (see e.g., [EtK]) and have been studied by a number of authors; see Dawson [Da], Dynkin [Dy1–4], El-Karoui and Roelly-Coppoletta [EK–RC, RC], Iscoe [I1,2], Jirina [J], Konno and Shiga [KS], Perkins [Pe], Watanabe [W] and the references therein.

Our interest lies in a special class of such processes, which may be loosely

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described as follows. At time zero, αN independent particles are set loose in E, each with initial law μ. The particles move independently according to the law of some Markov process ξ with values in E. A given particle lives an exponential amount of time — the mean lifetime being β_N — and upon its death gives birth to a random number of offspring. The offspring wander and propagate in the same fashion. Offspring are born at the death site of their parent, and the distribution $(p_k^{(N)}(x): k \geqq 0)$ of the number of offspring is allowed to depend on the death site x, and on the parameter N. The collection of particles alive at time t may be viewed as a random measure $X_t^{(N)}$ (each particle being given weight $1/N$). Under suitable hypotheses this sequence of measure-valued processes converges in distribution, as $N \to \infty$, to a limit measure-valued Markov branching process X with $X_0 = \alpha\mu$. Typically, $\beta_N \to 0$, $\Sigma_{k=1}^{\infty} kp_k^{(N)}(x) = 1 - \beta_N b_N(x)$, where $b_N(x) \to b(x)$, and

$$\lim_{N\to\infty}\left[(1-\lambda/N) - \sum_{k=0}^{\infty} p_k^{(N)}(x)(1-\lambda/N)^k\right](N/\beta_N) = \phi(x,\lambda),$$

where ϕ has the form

$$\phi(x,\lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda u} - \lambda u)n(x, du). \tag{1.2}$$

Here $c \geqq 0$ and b are bounded measurable functions and $n(x, du)$ is a kernel from $(E, \mathscr{E})$ to $(]0,\infty[, \mathscr{B}_{]0,\infty[})$. In view of (1.1) the infinitesimal generator L of X should preserve the class of exponentials $\mu \mapsto \exp(-\int f(x)\mu(dx))$. This, combined with (1.2), leads through a formal calculation to L:

$$\begin{aligned} L(F)(\mu) = \int_E \mu(dx)c(x)F''(\mu;x) + \int_E \mu(dx)[AF'(\mu;\cdot)(x) - b(x)F'(\mu;x)] \\ + \int_E \mu(dx)\int_0^\infty n(x,du)[F(\mu+u\varepsilon_x) - F(\mu) - uF'(\mu;x)]. \end{aligned} \tag{1.3}$$

Here A is the infinitesimal generator of the "spatial motion" process ξ, and $F'(\mu; x)$ and $F''(\mu; x)$ are the first and second variational derivatives of F (e.g., $F'(\mu;x) = \lim_{\delta\downarrow 0}(F(\mu+\delta\cdot\varepsilon_x) - F(\mu))/\delta$).

We shall not be concerned with limit theorems but rather with the construction and regularity properties of a wide class of "limit processes" having generators of the form (1.3). Previous authors have required that the coefficients b, c, and n be continuous in x, and that ξ possess a Feller continuous semigroup. We shall show that under much weaker hypotheses ($b, c \geqq 0$, and n suitably bounded and measurable; ξ a Borel right process) the process X

associated with L exists and is quite regular. Indeed X is always a Borel right process with quasi-left continuous natural filtration, and if ξ is a Hunt process then so is X. As might be expected from the form (1.3) of L, X has continuous paths (in a suitable topology on $M(E)$) if and only if $n = 0$. (This result can be found in [EK–RC] under the "Feller" hypotheses mentioned above; see also Dynkin [Dy1] for a related result.)

Section 2 contains our basic existence theorem, and further regularity results are obtained in Section 3, as is a description of the entrance space of X. Since we make no Feller continuity hypotheses, the theory of Ray processes plays an important role in these matters. In Section 4 we make precise the sense in which L is the generator of X, and we consider the path-continuity of X. Section 5 contains a brief discussion of a slightly wider class of processes whose study can be reduced to that of the class considered in the body of the paper. In an appendix we collect certain facts concerning Laplace transforms that are relevant to the discussion in Section 2.

Notation. For the most part we use the standard notation of Markov process theory; see [BG] and [Sh]. A few specifics are noted in what follows. Let $(E, \mathscr{E})$ be a measurable space. We write $b\mathscr{E}$ (resp. $p\mathscr{E}$, resp. $bp\mathscr{E}$) for the class of bounded (resp. positive, resp. bounded and positive) real-valued $\mathscr{E}$-measurable functions on E. The space of finite measures on $(E, \mathscr{E})$ is denoted $M(E)$, and $\mathscr{M}(E)$ is the σ-field on $M(E)$ generated by the mappings

$$l_f: \mu \mapsto \langle \mu, f \rangle, \qquad f \in bp\mathscr{E},$$

where $\langle \mu, f \rangle := \int_E f \, d\mu$. Likewise, the exponential mapping

$$\mu \mapsto \exp(-\langle \mu, f \rangle)$$

is denoted e_f. Note that if E is, for example, a topological Lusin space (i.e., homeomorphic to a Borel subset of a compact metric space) with Borel sets $\mathscr{E}$, then $\mathscr{M}(E)$ is the Borel σ-field on $M(E)$ endowed with the concomitant weak* topology. If F is a topological space, then $C(F)$ denotes the space of real-valued continuous functions on F; if F is locally compact and second countable, then $C_0(F)$ denotes those elements of $C(F)$ that tend to zero at infinity.

2. Basic construction

Throughout Sections 2 through 4 we work with a fixed Borel right Markov process $\xi = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, \xi_t, P^x)$, with Lusin state space $(E, \mathscr{E})$, semigroup (P_t), and resolvent (U^α). We assume that $P_t 1 = 1$ so that ξ has infinite lifetime.

We also fix a "branching mechanism" ϕ of the form (1.2), where $c \geqq 0$ and b are bounded and $\mathscr{E}$-measurable, and n is a (positive) kernel from $(E, \mathscr{E})$ to $(]0, \infty[, \mathscr{B}_{]0,\infty[})$ satisfying $\int_0^\infty (u \vee u^2) n(\cdot, du) \in b\mathscr{E}$. (This "finite variance" hypothesis is stronger than is really needed, but it serves to simplify the discussion.) Clearly $\phi(x, \cdot) \in C^2(\mathbf{R}_+)$ for all $x \in E$. Following Dynkin [Dy1–4], we shall refer to the measure-valued process X to be associated with ξ and ϕ as a *(ξ, ϕ)-superprocess*.

Formal computations using (1.1) and (1.3) suggest that X should have transition semigroup (Q_t) determined by its action on $\{e_f : f \in bp\mathscr{E}\}$; namely,

$$Q_t(\mu, e_f) = \exp(-\langle \mu, V_t f \rangle), \tag{2.1}$$

for all $\mu \in M(E)$, $t \geqq 0$, and $f \in bp\mathscr{E}$, where $(V_t : t \geqq 0)$ is the solution of the integral equation

$$v_t(x) = P_t f(x) + \int_0^t P_s(x, \phi(\cdot, v_{t-s}))ds. \tag{2.2}$$

Note that if A denotes the infinitesimal generator of ξ, then (2.2) is formally equivalent to

$$dv_t(x)/dt = Av_t(x) + \phi(x, v_t(x)), \qquad v_0 = f. \tag{2.2$'$}$$

Formula (2.1) amounts to a specification of the *Laplace transform* of the probability measure $Q_t(\mu, \cdot)$; that the R.H.S. of (2.1) is indeed a Laplace transform follows from the result (A.6) stated in the appendix once we observe that $f \mapsto V_t f(x)$ is negative definite on the semigroup $bp\mathscr{E}$ (the semigroup operation being pointwise addition). Recall that a real-valued function ψ on an Abelian semigroup $(S, +)$ is *negative definite* provided

$$\sum_{i,j} a_i a_j \psi(s_i + s_j) \leqq 0$$

whenever $\{a_1, \dots, a_n\} \subset \mathbf{R}$ sum to zero, $\{s_1, \dots, s_n\} \subset S$, and $n \geqq 2$. Clearly ϕ is negative definite on $(\mathbf{R}_+, +)$, and consequently the solution of (2.1) is negative definite. The following summary treats a slight extension of (2.2). The proof is a standard application of the method of Picard iteration (note that $\lambda \mapsto \phi(x, \lambda)$ is locally Lipschitz on $\mathbf{R}_+$, uniformly in $x \in E$). For discussions of integral equations of the specific type (2.2) see [D1; I1,2; W]; as a general reference see Pazy [Pa].

(2.3) Proposition. (a) *Given $f, g \in bp\mathscr{E}$, the equation*

(2.4) $$w_t(x) = P_t f(x) + \int_0^t P_s g(x)ds + \int_0^t P_s(x, \phi(\cdot, w_{t-s}))ds, \quad t \geqq 0,\ x \in E,$$

has a unique solution $(t, x) \mapsto W_t(f, g)(x)$ *jointly measurable in* (t, x) *and such that* $\sup_{0 \leqq s \leqq t} \| w_s \|_\infty < \infty$ *for all* $t > 0$.

(b) *If* $\beta = \| b^- \|_\infty$, *then*

$$0 \leqq W_t(f, g)(x) \leqq e^{\beta t} \| f \|_\infty + (e^{\beta t} - 1) \| g \|_\infty / \beta, \qquad \forall t \geqq 0, \quad x \in E.$$

(c) *If* $t \mapsto P_t f(x)$ *is right continuous on* $\mathbf{R}_+$ *for all* $x \in E$, *then so is* $t \mapsto W_t(f, g)(x)$.

(d) *If* $g \in bp\mathscr{E}$ *is fixed, then the mappings* $f \mapsto W_t(f, g)$, $t \geqq 0$, *form a nonlinear semigroup of operators on* $bp\mathscr{E}$.

(e) *For all* $t \geqq 0$ *and* $\mu \in M(E)$, *the mapping* $(f, g) \mapsto \langle \mu, W_t(f, g) \rangle$ *is negative definite on the semigroup* $bp\mathscr{E} \times bp\mathscr{E}$.

(2.5) Remarks. (a) Proposition (2.3) and its Corollary (2.6) to follow are valid for any measurable semigroup of positive sub-Markovian contractions on $b\mathscr{E}$.

(b) If $\tilde{\phi}$ is a second function of the form (1.2) let $N := \{x \in E : \phi(x, \cdot) \neq \tilde{\phi}(x, \cdot)\}$; suppose that $U^1 1_N = 0$. In this case it is easy to check that if $\tilde{W}_t(f, g)$ is the solution of (2.4) with ϕ replaced by $\tilde{\phi}$, then $\tilde{W}_t(f, g)(x) = W_t(f, g)(x)$ for all $t \geqq 0$ and $x \in E$.

Since (2.4) has unique solutions, if $f_n \downarrow f$ and $g_n \downarrow g$, then $W_t(f_n, g_n) \downarrow W_t(f, g)$ by the dominated convergence theorem, the monotonicity of $W_t(f_n, g_n)$ following from (A.5). Coupled with a standard result (A.6) on Laplace transforms recalled in the appendix, this yields the following

(2.6) Corollary. *For each* $t \geqq 0$ *and* $\mu \in M(E)$ *there exists a unique subprobability measure* $\Pi_t(\mu; \cdot)$ *on* $(M(E), \mathscr{M}(E))^2$ *such that*

$$\int \Pi_t(\mu; d\nu, d\gamma) e_f(\nu) e_g(\gamma) = \exp(-\langle \mu, W_t(f, g) \rangle)$$

for all $f, g \in bp\mathscr{E}$. *Moreover,* $\mu \mapsto \Pi_t(\mu; e_f \otimes e_g)$ *is* $\mathscr{M}(E)$*-measurable for all such* f, g.

Taking $g = 0$ in (2.3) we obtain the solution $V_t f = W_t(f, 0)$ of (2.2); since $V_t(0) = 0$, (2.6) guarantees the existence of *Markov* kernels Q_t, $t \geqq 0$, on $(M(E), \mathscr{M}(E))$ such that (2.1) holds. Indeed, $Q_t(\mu, e_f) = \Pi_t(\mu; e_f \otimes 1)$. In view of (2.3d) and the uniqueness theorem (A.6) for Laplace transforms, $(Q_t : t \geqq 0)$ is a semigroup on $b\mathscr{M}(E)$. Our goal in the rest of this section, and the next, is to prove that (Q_t) is the semigroup of a "nice" right process X. (See Section 5 for the interpretation of Π_t in terms of X.) The following result is the key tool in our analysis.

(2.7) PROPOSITION. *For each $t \geqq 0$ and $\mu \in M(E)$, the first two "moments" of $Q_t(\mu, \cdot)$ are finite and are given for $f \in bp\mathscr{E}$ by*

$$\int Q_t(\mu, d\nu) l_f(\nu) = \langle \mu, P_t^b f \rangle,$$

$$\int Q_t(\mu, d\nu) l_f(\nu)^2 = \langle \mu, P_t^b f \rangle^2 + \int_0^t \mu P_s(\hat{c} \cdot (P_{t-s}^b f)^2) ds,$$

where $\hat{c} = 2c + \int_0^\infty u^2 n(\cdot, du) \in bp\mathscr{E}$, and (P_t^b) is the semigroup on $b\mathscr{E}$ determined by

$$P_t^b f(x) = P^x\left(\exp\left(-\int_0^t b(\xi_s) ds\right) f(\xi_t)\right). \tag{2.8}$$

REMARK. Since b may be signed, P_t^b need not be sub-Markovian, but clearly $\| P_t^b f \|_\infty \leqq e^{\beta t} \| f \|_\infty$ where β is as in (2.3b).

PROOF. Since $V_t(\cdot)$ is $\mathbf{R}_+$-valued and negative definite, $\lambda \mapsto V_t(\lambda f)$ is an increasing, concave function on $\mathbf{R}_+$; see (A.5). By (2.3) and dominated convergence, the limit

$$k_t := \uparrow \lim_{\lambda \downarrow 0} \lambda^{-1} V_t(\lambda f) \leqq e^{\beta t} \| f \|_\infty < \infty$$

is the unique solution of

$$k_t = P_t f + \int_0^t P_s(b k_{t-s}) ds, \qquad t \geqq 0. \tag{2.9}$$

But $P_t^b f$ is also a solution of (2.9), hence $k_t = P_t^b f$. This and (2.1) yield the first moment of $Q_t(\mu, \cdot)$. The second moment identity follows by similar means, using the evaluation $d^2\phi(x, \lambda)/d\lambda^2 \,|_{\lambda=0} = -\hat{c}(x)$; cf. Dynkin [D2]. □

Before turning to the construction of a Markov process with semigroup (Q_t) we need some additional notation. In lieu of "Feller" hypotheses, we make heavy use of compactification methods. Good references for this material are the books of Getoor [G1] and Sharpe [Sh]. Let $\mathscr{R} \subset bp\mathscr{E}$ be a *countable* Ray cone for ξ constructed as in [Sh, §17]; since $\alpha U^\alpha 1 = 1$, we can and do assume that each function in $\mathscr{R}$ is bounded away from 0. Let $(\bar{E}, \bar{\mathscr{E}})$ be the corresponding Ray–Knight compactification of $(E, \mathscr{E})$, and let $(\bar{P}_t)$ and $(\bar{U}^\alpha)$ denote the associated extensions of (P_t) and (U^α). We write $\bar{\xi} = (\bar{\xi}_t, \bar{P}^x)$ for the canonical Ray process with semigroup $(\bar{P}_t)$.

Since E is Lusin and (P_t) is Borel, $E \in \bar{\mathscr{E}}$ and $\mathscr{E}$ is the trace of $\bar{\mathscr{E}}$ on E. These facts in hand we extend b, c, n, and ϕ trivially to $\bar{E}$ (setting them equal to zero on $\bar{E} \setminus E$) to obtain a function $\bar{\phi}$ on $\bar{E} \times \mathbf{R}_+$ of the form (1.2). Applying (2.3) and (2.6) to $(\bar{P}_t)$ and $\bar{\phi}$ we obtain operators $\bar{W}_t$, $\bar{V}_t$, and a semigroup $(\bar{Q}_t)$ on $M(\bar{E})$. The analogue of (2.7) is valid if $\bar{P}_t^b$ is defined by the R.H.S. of (2.8) with P^x, ξ, and b replaced by $\bar{P}^x$, $\bar{\xi}$, and $\bar{b}$.

We identify $M(E)$ with the subset of $M(\bar{E})$ comprising those elements of $M(\bar{E})$ that are carried by E, and then $M(E) \in \mathscr{M}(\bar{E})$ and $\mathscr{M}(E) = \mathscr{M}(\bar{E}) \cap M(E)$. In the same way, any measure on $(M(E), \mathscr{M}(E))$ can be regarded as a measure on $(M(\bar{E}), \mathscr{M}(\bar{E}))$ which is carried by $M(E)$. Since $\varepsilon_x \bar{P}_t \,|_E = \varepsilon_x P_t$ if $x \in E$, we have $\bar{V}_t(x, \bar{f}) = V_t(x, f)$ if $x \in E$, where $\bar{f} \in bp\bar{\mathscr{E}}$ and $f = \bar{f}\,|_E$, because of the uniqueness assertion in (2.3a). The uniqueness in (2.6) now forces

$$(2.10) \qquad \varepsilon_\mu \bar{Q}_t \,|_{M(E)} = \varepsilon_\mu Q_t, \qquad \mu \in M(E).$$

Moreover, for $t \geqq 0$ and $\mu \in M(\bar{E})$, we have $\bar{Q}_t(\mu\bar{P}_0, \cdot) = \bar{Q}_t(\mu, \cdot)$. We shall show that for each $\mu \in M(E)$ there is a càdlàg $M(\bar{E})$-valued Markov process X with semigroup $(\bar{Q}_t)$ and $X_0 = \mu$. This being done we show that X actually spends all of its time in $M(E)$, and in the process that X has the strong Markov property. In view of (2.10) it will then follow that X has semigroup (Q_t), and so is the desired (ξ, ϕ)-superprocess.

Since $\bar{E}$ is a compact metric space, $M(\bar{E})$ endowed with its weak* topology is a locally compact separable metric space. We refer to the relative topology on $M(E)$ as the *weak Ray topology*, and write $M_r(E)$ for this topological space. In contrast, we write $M_o(E)$ to indicate $M(E)$ with the weak topology induced by the mappings l_f as f runs through the bounded continuous functions on E with its original topology. Clearly $\mathscr{M}(\bar{E})$ is the Borel σ-field on $M(\bar{E})$, and the Borel σ-fields on $M_r(E)$ and $M_o(E)$ both coincide with $\mathscr{M}(E)$.

Let $D := \{x \in \bar{E} : \bar{P}_0(x, \cdot) = \varepsilon_x\}$ and $B := \bar{E} \setminus D$ denote the non-branch points and branch points of $(\bar{P}_t)$; both of these sets are $\bar{\mathscr{E}}$-measurable. Let W_0 denote the space of right continuous paths from $\mathbf{R}_+$ into $M(D)$ with left limits in $M(\bar{E})$. (Of course $M(D)$ carries the relative topology inherited from $M(\bar{E})$.) We write $X = (X_t : t \geqq 0)$ for the coordinate process on W_0, and put $\mathscr{G}^\circ = \sigma\{X_t : t \geqq 0\}$. Finally, each function f in the Ray cone $\mathscr{R}$ admits a unique extension by continuity to $\bar{E}$; we write $\bar{f}$ for this extension and put

$\bar{\mathscr{R}} = \{\bar{f} : f \in \mathscr{R}\}$. Our construction of the (ξ, ϕ)-superprocess begins with the following.

(2.11) THEOREM. *Given $\mu \in M(E)$ there exists a unique probability measure $\mathbf{P}_\mu$ on $(W_0, \mathscr{G}^\circ)$ such that $\mathbf{P}_\mu(X_0 = \mu) = 1$ and $(X_t : t \geqq 0)$ under $\mathbf{P}_\mu$ is a Markov process with semigroup $(\bar{Q}_t)$.*

PROOF. Let $\Omega = M(\bar{E})^{\mathbf{R}_+}$, $Z_t(\omega) = \omega(t)$, and $\mathscr{A} = \sigma\{Z_t : t \geqq 0\}$. Fix $\mu \in M(E)$. Kolmogorov's theorem guarantees a unique probability measure $\mathbf{P}$ on $(\Omega, \mathscr{A})$ such that $\mathbf{P}(Z_0 = \mu) = 1$ and under which (Z_t) is a Markov process with semigroup $(\bar{Q}_t)$. Note that $\langle Z_t, 1_{\bar{E}\setminus E}\rangle = 0$ a.s. $\mathbf{P}_\mu$ for all $t \geqq 0$. Let $\beta = \| b^- \|_\infty$. Each $\bar{f} \in \bar{\mathscr{R}}$ is α-supermedian relative to $(\bar{P}_t)$ for some $\alpha = \alpha(f) > 0$. In this case, by (2.7), $e^{-(\alpha+\beta)t}\langle Z_t, \bar{f}\rangle$ is a positive $\mathbf{P}$-supermartingale. Also, if $\gamma = \| b^+ \|_\infty$, then by (2.7), $e^{\gamma t}\langle Z_t, 1_{\bar{E}}\rangle$ is a $\mathbf{P}$-submartingale. Doob's inequality now yields

$$\mathbf{P}\left(\sup_{0 \leqq r \leqq t,\, r \text{ rat.}} \langle Z_r, 1_{\bar{E}}\rangle^2\right) \leqq 4e^{2\gamma t}\mathbf{P}(\langle Z_t, 1_{\bar{E}}\rangle^2) < \infty, \tag{2.12}$$

and by a standard supermartingale argument we see that if Ω_0 denotes the set of $\omega \in \Omega$ such tht $t \mapsto \langle Z_t(\omega), \bar{f}\rangle$ has left and right limits along the rationals at each $t \geqq 0$ for all $\bar{f} \in \bar{\mathscr{R}}$, and such that $\sup_{0 \leqq r \leqq t,\, r \text{ rat.}}\langle Z_r(\omega), 1_{\bar{E}}\rangle < \infty$ for all $t > 0$, then Ω_0 lies in the $\mathbf{P}$-completion of $\mathscr{A}$ and $\mathbf{P}(\Omega_0) = 1$. Define for $t \geqq 0, \bar{f} \in \bar{\mathscr{R}}$,

$$Z_{t+}(\omega, \bar{f}) = \begin{cases} 0, & \text{if } \omega \in \Omega \setminus \Omega_0, \\ \lim_{r \downarrow t,\, r \text{ rat.}} \langle Z_r(\omega), \bar{f}\rangle, & \text{if } \omega \in \Omega_0. \end{cases}$$

Since $\bar{\mathscr{R}} \subset bpC(\bar{E})$ separates the points of $\bar{E}$ and each $\bar{f} \in \bar{\mathscr{R}}$ is bounded away from 0, the Stone–Weierstrass theorem implies that $\{e_{\bar{f}} : \bar{f} \in \bar{\mathscr{R}}\}$ is total in $C_0(M(\bar{E}))$. Because of the right continuity of $t \mapsto \bar{P}_t\bar{f}(x)$, (2.3c) and (2.1) show that $t \mapsto \bar{Q}_t(\nu, e_{\bar{f}})$ is right continuous on $[0, \infty[$ for all $\nu \in M(\bar{E})$, $\bar{f} \in \bar{\mathscr{R}}$. It follows that $t \mapsto \bar{Q}_t(\nu, F)$ is right continuous for all $F \in C_0(M(\bar{E}))$, hence that (Z_t) is right continuous in probability under $\mathbf{P}$. Thus

$$\mathbf{P}(\langle Z_t, \bar{f}\rangle = Z_{t+}(\bar{f}),\ \forall \bar{f} \in \bar{\mathscr{R}}) = 1, \qquad \forall t \geqq 0. \tag{2.13}$$

Clearly $Z_{t+}(\cdot)$ extends uniquely to a positive linear functional on $\bar{\mathscr{R}} - \bar{\mathscr{R}}$ satisfying

$$|Z_{t+}(\omega, \bar{f})| \leqq \|\bar{f}\|_\infty \cdot Z_{t+}(\omega, 1_{\bar{E}}) < \infty.$$

Since $\bar{\mathscr{R}} - \bar{\mathscr{R}}$ is uniformly dense in $C(\bar{E})$, there exists a unique random variable $\omega \mapsto Z_{t+}(\omega) \in M(\bar{E})$ such that

$$\langle Z_{t+}(\omega), \bar{f} \rangle = Z_{t+}(\omega, \bar{f}), \qquad \forall \bar{f} \in \bar{\mathscr{R}},$$

identically in $t \geqq 0$ and $\omega \in \Omega$. It is clear that (Z_{t+}) is a càdlàg $M(\bar{E})$-valued Markov process with semigroup $(\bar{Q}_t)$, and $Z_{0+} = \mu$ a.s. $\mathbf{P}$. We claim that

$$\mathbf{P}(\langle Z_{t+}, 1_B \rangle = 0, \ \forall t \geqq 0) = 1. \tag{2.14}$$

Since $D = \bar{E} \setminus B$, once (2.14) is verified we can let $\mathbf{P}_\mu$ denote the image of $\mathbf{P}$ under the mapping $\omega \mapsto Z_{\cdot +}(\omega) \in W_0$, and the theorem will be proved. To see (2.14) recall from [Sh, (9.11)] that

$$B = \{x \in \bar{E} : \bar{f}(x) > \bar{P}_0 \bar{f}(x) \text{ for some } \bar{f} \in \bar{\mathscr{R}}\};$$

thus it suffices to show that $\mathbf{P}(\langle Z_{t+}, \bar{f} \rangle = \langle Z_{t+}, \bar{P}_0 \bar{f} \rangle, \ \forall t \geqq 0) = 1$ for all $\bar{f} \in \bar{\mathscr{R}}$. (Each $\bar{f} \in \bar{\mathscr{R}}$ is α-supermedian relative to $(\bar{P}_t)$ for some $\alpha > 0$, so $\bar{f} \geqq \bar{P}_0 \bar{f}$.) Fix $\bar{f} \in \bar{\mathscr{R}}$; now $\bar{f} = \bar{P}_0 \bar{f}$ on E so by (2.13) we have $\langle Z_{t+}, \bar{f} \rangle = \langle Z_{t+}, \bar{P}_0 \bar{f} \rangle$ a.s. $\mathbf{P}$, $\forall t \geqq 0$. Next, $t \mapsto \langle Z_{t+}, \bar{f} \rangle$ is right continuous by construction. On the other hand, there is an $\alpha > 0$ and a sequence $(g_n) \subset C(\bar{E})$ such that $\bar{U}^\alpha g_n \uparrow \bar{P}_0 \bar{f}$ (see [G1, p. 21]). Since $\bar{U}^\alpha(C(\bar{E})) \subset C(\bar{E})$, $t \mapsto e^{-(\alpha+\beta)t} \langle Z_{t+}, \bar{U}^\alpha g_n \rangle$ is a positive right continuous supermartingale for each n. But a theorem of P.-A. Meyer asserts that the increasing limit of a sequence of right continuous supermartingales is itself right continuous almost surely. Thus $t \mapsto \langle Z_{t+}, \bar{P}_0 \bar{f} \rangle$ is right continuous a.s. $\mathbf{P}$, and so $\mathbf{P}(\langle Z_{t+}, \bar{f} \rangle = \langle Z_{t+}, \bar{P}_0 \bar{f} \rangle, \ \forall t \geqq 0) = 1$, which yields (2.14). □

Given $\mu \in M(E)$ let $(W_0, \mathscr{G}^\circ, \mathbf{P}_\mu)$ and $X = (X_t)$ be as provided by Theorem (2.11). Let $\mathscr{G}_t^\circ = \sigma\{X_s : 0 \leqq s \leqq t\}$ and let $(\mathscr{G}_t^\mu)$ denote the $\mathbf{P}_\mu$-augmentation of $(\mathscr{G}_{t+}^\circ)$. The reader familiar with [G1] or [Sh] will note that the latter part of the proof of (2.11) parallels a standard argument in the theory of Ray processes. In the same way one can adapt the proofs of [G1, (5.8), (5.11)] to obtain the following "linear" versions of the strong and moderate Markov properties of X. Note that as a consequence of (2.12) and the right continuity of X, if T is a *bounded* $(\mathscr{G}_t^\mu)$ stopping time, then

$$\mathbf{P}_\mu(\langle X_T, 1_E \rangle^2) < \infty.$$

We write X_{t-}^r for the left limit of X in $M(\bar{E})$ at time $t > 0$.

(2.15) PROPOSITION. *Fix $\mu \in M(E)$, $\bar{f} \in b\bar{\mathscr{E}}$, $s \geqq 0$, and let T be a bounded $(\mathscr{G}_t^\mu)$ stopping time. Then*

$$\mathbf{P}_\mu(\langle X_{T+s}, \bar{f}\rangle \mid \mathcal{G}_T^\mu) = \langle X_T, \bar{P}_s^b \bar{f}\rangle.$$

If, in addition, T is $\mathbf{P}_\mu$-predictable, then

$$\mathbf{P}_\mu(\langle X_{T+s}, \bar{f}\rangle \mid \mathcal{G}_{T-}^\mu) = \langle X_{T-}^r, \bar{P}_s^b \bar{f}\rangle.$$

Before proceeding to the full strong Markov property of X, we use (2.15) to show that X is actually $M(E)$-valued. The argument that follows was inspired by recent work of J. Steffens [St].

(2.16) COROLLARY. *For all $\mu \in M(E)$,*

$$\mathbf{P}_\mu(\langle X_t, 1_{\bar{E}\setminus E}\rangle = 0 \textit{ for all } t \geqq 0) = 1.$$

PROOF. It suffices to show that $\langle X_\cdot, 1_{D\setminus E}\rangle$ is $\mathbf{P}_\mu$-evanescent, for all $\mu \in M(E)$. Fix μ and let T be a bounded $(\mathcal{G}_t^\mu)$ stopping time. Choose $\alpha > \| b^- \|_\infty$ and define a *bounded* kernel $U^{\alpha+b}$ by

$$U^{\alpha+b}f(x) = P^x \int_0^\infty \exp\left(-\alpha t - \int_0^t b(\xi_s)ds\right) f(\xi_t)dt.$$

Let $\bar{U}^{\alpha+b}$ be defined analogously. Define $\nu \in M(D)$ by

$$\nu(f) = \mathbf{P}_\mu(e^{-\alpha T}\langle X_T, f\rangle).$$

Then by (2.15),

$$\begin{aligned}
\mu U^{\alpha+b} f &= \mathbf{P}_\mu\left(\int_0^\infty e^{-\alpha t}\langle X_t, f\rangle dt\right) \\
&\geqq \mathbf{P}_\mu\left(e^{-\alpha T}\int_0^\infty e^{-\alpha s}\langle X_{T+s}, f\rangle ds\right) \\
&= \mathbf{P}_\mu(e^{-\alpha T}\langle X_T, \bar{U}^{\alpha+b} f\rangle) = \nu \bar{U}^{\alpha+b} f.
\end{aligned}$$

It follows that $\nu\bar{U}^{\alpha+b}$ is carried by E and is an excessive measure relative to the *right* semigroup $(P_t^{\alpha+b})$. By [GG, (4.2)], the inequality $\mu U^{\alpha+b} \geqq \nu\bar{U}^{\alpha+b}$ implies the existence of a unique $\gamma \in M(E)$ such that $\nu\bar{U}^{\alpha+b} = \gamma U^{\alpha+b}$. Regarding γ as a measure on D that is carried by E, we see that $\gamma\bar{U}^{\alpha+b} = \nu\bar{U}^{\alpha+b}$. But $\bar{U}^{\alpha+b}$ restricted to D is the 0-potential of a right process so we may invoke uniqueness of charges [GG, (1.1)] to deduce that $\gamma = \nu$. Since γ is carried by E, $\mathbf{P}_\mu(\langle X_T, 1_{D\setminus E}\rangle) = 0$; since T was an arbitrary bounded stopping time, an application of the optional section theorem finishes the proof. □

Without otherwise altering our notation, we now take the sample space of X

to be W_1, the space of right continuous paths from $\mathbf{R}_+$ into $M_r(E)$ with left limits in $M(\bar{E})$. (Recall that $M_r(E)$ is $M(E)$ with the weak Ray topology.)

(2.17) THEOREM. *$X=(X_t, \mathbf{P}_\mu)$ is an $M_r(E)$-valued Borel right process with semigroup (Q_t). Moreover, for $F\in b\mathcal{M}(\bar{E})$, $s\geqq 0$, and any $(\mathcal{G}_t^\mu)$ predictable time T,*

$$\mathbf{P}_\mu(F(X_{T+s})\,|\,\mathcal{G}_{T-}^\mu)=\bar{Q}_sF(X_{T-}^r), \qquad a.s.\ \mathbf{P}_\mu \text{ on } \{T<\infty\}. \tag{2.18}$$

PROOF. Since $M_r(E)$ is Lusin and (Q_t) is Borel measurable, in showing that X is a right process it is enough to check the strong Markov property:

$$\mathbf{P}_\mu(F(X_{T+s})\,|\,\mathcal{G}_T^\mu)=Q_sF(X_T) \tag{2.19}$$

a.s. $\mathbf{P}_\mu$ on $\{T<\infty\}$. Fix $\mu\in M(E)$ and $\bar{f}\in bp\bar{\mathcal{E}}$. By martingale theory, the first identity in (2.15) implies that for all $t>0$, $s\mapsto\langle X_s, \bar{P}_{t-s}^b\bar{f}\rangle$ is a right continuous $(\mathcal{G}_s^\mu)$ martingale on $[0,t[$. Now $\bar{V}_t\bar{f}$, being the solution of the "barred" version of (2.2), is also the unique solution of

$$\bar{V}_t\bar{f}(x)=\bar{P}_t^b\bar{f}(x)+\int_0^t\bar{P}_s^b(x,\bar{\psi}(\cdot,\bar{V}_{t-s}\bar{f}))ds, \qquad t\geqq 0,$$

where $\bar{\psi}(x,\lambda)=\bar{\phi}(x,\lambda)+\bar{b}(x)\lambda$. It follows that $s\mapsto\langle X_s, \bar{V}_{t-s}\bar{f}\rangle$ is right continuous on $[0,t[$ a.s. $\mathbf{P}_\mu$. In view of (2.1) and the totality of $\{e_{\bar{f}}:\bar{f}\in\bar{\mathcal{R}}\}$ in $C_0(M(\bar{E}))$, $s\mapsto\bar{Q}_{t-s}F(X_s)$ is right continuous on $[0,t[$ a.s. $\mathbf{P}_\mu$ for all $t>0$, $F\in C_0(M(\bar{E}))$. This implies that X is strong Markov with semigroup $(\bar{Q}_t)$; see [Sh, (7.4)]. Corollary (2.16) now finishes the proof of (2.19), since $Q_t=\bar{Q}_t\,|_{M(E)}$. Using the second identity in (2.15) and the left-handed version of the above argument one obtains (2.18). □

In fact, X is a Hunt process when viewed as a process with values in the topological space $M_r(E)$:

(2.20) THEOREM. *For all $\mu\in M(E)$.*

$$\mathbf{P}_\mu(\langle X_{t-}^r, 1_{\bar{E}\setminus E}\rangle=0,\ \forall t>0)=1;$$

that is, $X_-^r=(X_{t-}^r:t>0)$ is $M_r(E)$-valued a.s. $\mathbf{P}_\mu$. Moreover, $(\mathcal{G}_t^\mu)$ is quasi-left continuous, and X is an $M_r(E)$-valued Hunt process.

PROOF. We claim that for all $\mu\in M(E)$ and all bounded $(\mathcal{G}_t^\mu)$ predictable times T, $X_T=\int X_{T-}^r(dx)\bar{P}_0(x,\cdot)$ a.s. $\mathbf{P}_\mu$. Indeed, for such T we have by the second identity in (2.15),

$$\text{(2.21)}\qquad \mathbf{P}_\mu([\langle X_T, \bar f\rangle - \langle X^r_{T-}, \bar P_0 \bar f\rangle]^2) = \mathbf{P}_\mu(\langle X_T, \bar f\rangle^2) - \mathbf{P}_\mu(\langle X^r_{T-}, \bar P_0 \bar f\rangle^2),$$

for all $\bar f \in \bar{\mathscr{R}}$, since $\bar P_0^b = \bar P_0$. By (2.18) and the "barred" version of the second moment identity in (2.7),

$$\mathbf{P}_\mu(\langle X_T, \bar f\rangle^2 \mid \mathscr{G}^\mu_{T-}) = \bar Q_0(X^r_{T-}, (l_{\bar f})^2) = \langle X^r_{T-}, \bar P_0 \bar f\rangle^2,$$

so the L.H.S. of (2.21) vanishes and the claim is verified. Thus $\mathscr{G}^\mu_T = \mathscr{G}^\mu_{T-}$ by [Sh, (23.8)], and an adaptation of the argument used in [Sh, (42.5i)] now shows that $X_T = X^r_{T-}$ a.s. $\mathbf{P}_\mu$ on $\{0 < T < \infty\}$ for any $(\mathscr{G}^\mu_t)$ predictable time T. The predictable set $\{t > 0 : X^r_{t-} \notin M(E)\}$ is therefore $\mathbf{P}_\mu$-evanescent, and the quasi-left continuity of X in $M_r(E)$ follows as in [Sh, (47.6)]. The remaining assertions are now evident. □

3. Further regularity

Our principle goal in this section is to show that the (ξ, ϕ)-superprocess X constructed in the last section has paths which are right continuous in $M_o(E)$ ($M(E)$ endowed with the weak* topology induced by the original topology on E). To this end we record criteria for the regularity of processes of the form $t \mapsto f(\xi_t)$ based on the balayage order. These results can then be "lifted" via (2.7) to yield the regularity of $t \mapsto \langle X_t, f\rangle$ for certain f.

Let $\tilde\xi = (\tilde\xi_t, \tilde P^x)$ be a Borel right process on $(E, \mathscr{E})$ with *bounded* potential kernel $\tilde U$. (In the application of what follows, $\tilde\xi$ will be a certain subprocess of ξ.) Since $\tilde U$ is bounded, the semigroup of $\tilde\xi$ is sub-Markovian and the lifetime $\tilde\zeta := \inf\{t : \tilde\xi = \partial\}$ of $\tilde\xi$ is finite a.s. If ∂ denotes the cemetery point for $\tilde\xi$, then any function f on E is extended to $E \cup \{\partial\}$ (the true state space of $\tilde\xi$) by setting $f(\partial) = 0$.

The *balayage order* $\dashv$ induced on $M(E)$ by $\tilde\xi$ is defined by

$$\text{(3.1)}\qquad \mu \dashv \nu \qquad \text{if and only if } \mu\tilde U \leqq \nu\tilde U.$$

Since $\tilde U$ is bounded, we have

$$\mu \dashv \nu \qquad \text{if and only if } [\mu(h) \leqq \nu(h),\ \forall\, h \in \mathscr{S}],$$

where $\mathscr{S}$ denotes the class of bounded $\tilde\xi$-excessive functions. Associated with $\dashv$ are two limit notions.

(3.2) Definition. Let ν and ν_n, $n \geqq 1$, be elements of $M(E)$. Then

$$\nu = \uparrow \text{bal–lim}\ \nu_n \ (\text{resp., } \nu = \downarrow \text{bal–lim}\ \nu_n)$$

$$\text{provided } \nu_n\tilde{U} \uparrow \nu\tilde{U} \text{ (resp., } \nu_n\tilde{U} \downarrow \nu\tilde{U})$$

setwise.

Clearly $\nu = \uparrow \text{bal–lim}\, \nu_n$ if and only if $\nu_n(h) \uparrow \nu(h)$ for all $h \in \mathscr{S}$. Similarly, $\nu = \downarrow \text{bal–lim}\, \nu_n$ if and only if $\nu_n(p) \downarrow \nu(p)$ for all bounded regular potentials, p, of ξ; this follows easily from [BG, IV(3.6)].

A *randomized stopping time* of ξ is a measurable family $\{T(u) : u \in \hat{\Omega}\}$ of stopping times of ξ, where $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ is an auxiliary probability space. By a theorem of Rost [R], if $\mu \dashv \nu$ then there is a randomized stopping time $\{T(u)\}$ such that for all $f \in bp\mathscr{E}$,

$$\mu(f) = \int_{\hat{\Omega}} \hat{P}(du)\tilde{P}^{\nu}(f(\xi_{T(u)}); T(u) < \zeta).$$

(In fact, one can always take $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ to be the unit interval equipped with Lebesgue measure.)

(3.3) PROPOSITION. *A function $f \in b\mathscr{E}$ is finely continuous (relative to ξ) if and only if $\nu_n(f) \to \nu(f)$ whenever $\nu = \uparrow$ bal–lim ν_n.*

PROOF. Assume that $\nu_n(f) \to \nu(f)$ whenever $\nu = \uparrow \text{bal–lim}\, \nu_n$. Fix $\mu \in M(E)$. Let (T_n) be a uniformly bounded decreasing sequence of stopping times of $(\xi_t, \tilde{P}^{\mu})$, with limit T. Define $\nu_n, \nu \in M(E)$ by $\nu_n(g) = \tilde{P}^{\mu}(g(\xi_{T_n}))$, $\nu(g) = \tilde{P}^{\mu}(g(\xi_T))$. Clearly $\nu = \uparrow \text{bal–lim}\, \nu_n$. The right continuity of $t \mapsto f(\xi_t)$ a.s. $\tilde{P}^{\mu}$ now follows from [DM1, VI.48a]. Since $\mu \in M(E)$ was arbitrary, f is finely continuous.

Conversely, let $f \in b\mathscr{E}$ be finely continuous and let $\nu = \uparrow \text{bal–lim}\, \nu_n$. By Rost's theorem, for each n there is a randomized stopping time $\{T_n(u); u \in [0, 1]\}$ such that

$$\nu_n(g) = \int_0^1 du\, \tilde{P}^{\nu}(g(\xi_{T_n(u)}); T_n(u) < \zeta), \qquad h \in b\mathscr{E}.$$

Since $\nu_n\tilde{U} \uparrow \nu\tilde{U}$,

$$\int_0^1 du\, \tilde{P}^{\nu}\left(\int_0^{T_n(u) \wedge \zeta} g(\xi_t)dt\right) \downarrow 0, \qquad n \to \infty,$$

for any strictly positive $g \in b\mathscr{E}$. Thus $T_n \to 0$ in probability (relative to $\text{Leb} \otimes \tilde{P}^{\nu}$). Since $t \mapsto f(\xi_t)$ is right continuous a.s. $\tilde{P}^{\nu}$, $f(\xi_{T_n}) \to f(\xi_0)$ in probability as well. The dominated convergence theorem now yields $\nu_n(f) \to \nu(f)$ as desired. □

For the sinister version of (3.3), let us say that $f \in b\mathscr{E}$ is *quasi-left continuous* (relative to ξ) if $f(\xi_{T_n}) \to f(\xi_T)$ a.s. on $\{T < \infty\}$ whenever (T_n) is an increasing sequence of ξ-stopping times with limit T. We write ${}^p f(\xi)$ for the predictable projection of the process $f(\xi)$.

(3.4) PROPOSITION. *Fix* $f \in b\mathscr{E}$.

(a) *The process* $t \mapsto f(\xi_t)$ *has left limits on* $]0, \infty[$ *a.s. if and only if* $\lim \nu_n(f)$ *exists whenever* $\downarrow$ bal–lim ν_n *exists in* $M(E)$.

(b) *If f is quasi-left continuous then* $\nu_n(f) \to \nu(f)$ *whenever* $\nu = \downarrow$ bal–lim ν_n. *Conversely, if the lifetime* ζ *of* ξ *is totally inaccessible and if* $\nu_n(f) \to \nu(f)$ *whenever* $\nu = \downarrow$ bal–lim ν_n, *then* $f(\xi)$ *has left limits on* $]0, \infty[$ *a.s., and the left limit process* $\{f(\xi)_{t-}, t > 0\}$ *is* $\mathbf{P}_\mu$*-indistinguishable from* ${}^p f(\xi)$ *for all* $\mu \in M(E)$.

PROOF. The proofs of (a) and (b) are quite similar, so we consider only (b), it being the more difficult point. Assume first that f is quasi-left continuous and that $\nu = \downarrow$ bal–lim ν_n. Using Rost's theorem it is not hard to construct a sequence $\{T_n(u) : u \in \hat{\Omega}\}_{n \geq 1}$ of randomized stopping times, where $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ is the product of countably many copies of the Lebesgue measure on $[0, 1]$, such that

(i) $T_n(\omega, u) \leqq T_{n+1}(\omega, u) \uparrow T(\omega, u)$, $\forall (\omega, u) \in \Omega \times \hat{\Omega}$,

(ii) $\nu_n = \int_{\hat{\Omega}} \hat{P}(du) \tilde{P}^{\nu_1}(\xi_{T_n(u)} \in \cdot\, ; T_n(u) < \zeta)$,

with the analog of (ii) relating ν to $T(u)$. By Fubini and the quasi-left continuity of f, $f(\xi_{T_n}) \to f(\xi_T)$ a.s. $\hat{P} \otimes \tilde{P}^{\nu_1}$ on $\{T < \infty\}$, so $\nu_n(f) \to \nu(f)$ by dominated convergence since $\zeta < \infty$ a.s. Conversely, suppose that ζ is totally inaccessible and that $\nu_n(f) \to \nu(f)$ whenever $\nu = \downarrow$ bal–lim ν_n. Arguing as in the proof of (3.3), it follows from [DM1, VI.48b] that $f(\xi)$ has left limits on $]0, \infty[$ a.s. A similar application of [DM1, VI.49] shows that ${}^p f(\xi)$ is left continuous on $]0, \infty[$ a.s. The desired conclusion now follows from [DM1, VI.50]. □

We now use Propositions (3.3) and (3.4) to deduce further continuity properties of X.

(3.5) THEOREM. *Fix* $f \in b\mathscr{E}$.

(a) *If f is finely continuous relative to* ξ, *then* $\langle X_\cdot, f \rangle$ *is right continuous on* $[0, \infty[$ *a.s. If* $f(\xi)$ *has left limits on* $]0, \infty[$ *a.s., then so does* $\langle X_\cdot, f \rangle$.

(b) *If f is quasi-left continuous relative to* ξ, *then* $l_f : \mu \mapsto \langle \mu, f \rangle$ *is quasi-left continuous relative to* X.

PROOF. (a) Fix $\mu \in M(E)$ and let (T_n) be a decreasing sequence of bounded $(\mathscr{G}_t^\mu)$ stopping times with limit T. Choose $\alpha > \| b^- \|_\infty$, define $\nu_n \in M(E)$ by

$$\nu_n(g) = \mathbf{P}_\mu(e^{-\alpha T_n}\langle X_{T_n}, g\rangle),$$

and define ν analogously with T replacing T_n. It is easy to check that $\nu = \uparrow$ bal–lim ν_n relative to the subprocess

$$\tilde{\xi} = \left(\xi, \exp\left(-\alpha t - \int_0^t b(\xi_s)ds\right)\right)$$

of ξ (cf. the proof of (2.16)). Note that $\tilde{\xi}$ has the bounded kernel $U^{\alpha+b}$ for its 0-potential kernel. Moreover, since the multiplicative functional $\exp(-\alpha t - \int_0^t b(\xi_s)ds)$ is continuous and strictly positive, ξ and $\tilde{\xi}$ have identical fine topologies and the lifetime of $\tilde{\xi}$ is totally inaccessible (the latter fact is needed for the second assertion in (a), and for (b)). By (3.3), $\nu_n(f) \to \nu(f)$. Since (T_n) was arbitrary, [DM1, VI.48] yields the almost sure right continuity of $e^{-\alpha t}\langle X_t, f\rangle$. The second assertion follows analogously from (3.4a).

(b) Fix $\mu \in M(E)$ and suppose that f is quasi-left continuous. Reasoning as in the proof of (a), it follows from (3.4b) that $t \mapsto \langle X_t, f\rangle$ has left limits a.s. $\mathbf{P}_\mu$ and that the left limit process is $\mathbf{P}_\mu$-indistinguishable from the $\mathbf{P}_\mu$-predictable projection of $\langle X., f\rangle$. By (2.17) and (2.20) this latter process is $\mathbf{P}_\mu$-indistinguishable from $\langle X^r_-, f\rangle$. But X is a Hunt process (when regarded as a process with values in $M_r(E)$), so $\{X^r_- \neq X\}$ meets the graph of no $(\mathscr{G}^\mu_t)$ predictable time. Thus $\langle X_T, f\rangle = \langle X, f\rangle_{T-}$ a.s. $\mathbf{P}_\mu$ for any $(\mathscr{G}^\mu_t)$ predictable time T, and the assertion follows. □

(3.6) COROLLARY. *The (ξ, ϕ)-superprocess X is a right process on the state space $M_o(E)$. If, in addition, ξ is a Hunt process, then so is X when viewed as an $M_o(E)$-valued process.*

We close this section by identifying the "entrance space" of ξ. Recall that a *probability entrance law* (for (P_t)) is a family $(\nu_t : t > 0)$ of probability measures on $(E, \mathscr{E})$ such that $\nu_t P_s = \nu_{t+s}$ for $s, t > 0$. Let $E_D = \{x \in D : \bar{U}^1(x, 1_E) = 1\} \in \bar{\mathscr{E}}$ denote the *entrance space* for ξ. It is well-known that each probability entrance law (ν_t) admits a unique representation $\nu_t = \nu_0 \bar{P}_t$, $t > 0$, where ν_0 is a probability measure on E_D. See [Sh, (40.2)]. Of course, the analogous result holds for the right process X, and we shall explicitly identify the entrance space of X as $M(E_D)$. See Dynkin [D3] for analogous results in the case $\phi(x, \lambda) =$ const. λ^2 but where ξ is allowed to be nonhomogeneous and X is allowed to take values in the space of σ-finite measures.

(3.7) THEOREM. *If $(N_t : t > 0)$ is a probability entrance law for (Q_t), then* *there is a unique probability law N_0 on $(M(E_D), \mathcal{M}(E_D))$ such that $N_t = N_0\bar{Q}_t$ for all $t > 0$.*

PROOF. Fix a probability entrance law (N_t) for (Q_t). Let W_+ denote the space of paths from $]0, \infty[$ into $M(E)$ that are càdlàg in $M_r(E)$ and right continuous in $M_o(E)$ on $]0, \infty[$. For this proof only let $(X_t : t > 0)$ be the coordinate process on W_+. Given a probability entrance law (N_t) for (Q_t), let **P** denote the unique probability measure on W_+ under which (X_t) is Markovian with semigroup (Q_t) and 1-dimensional distributions (N_t). (See [Sh, §40] for the existence of **P**). Let $(\mathcal{G}_t)$ denote the **P**-augmentation of the natural filtration (made right continuous) of (X_t).

For $t, \lambda > 0$, by the martingale convergence theorem

$$0 < \mathbf{P}(e^{-\lambda\langle X_t, 1\rangle} \mid \mathcal{G}_0) = \lim_{r\downarrow 0} \mathbf{P}(e^{-\lambda\langle X_t, 1\rangle} \mid \mathcal{G}_r)$$

$$= \lim_{r\downarrow 0} \exp - \langle X_r, V_{t-r}(\lambda 1_E)\rangle$$

a.s. **P**. It follows that

$$\lim_{r\downarrow 0} \langle X_r, V_{t-r}(\lambda 1_E)\rangle \quad \text{exists in } [0, \infty[,$$

a.s. **P** for all $t > 0$. By Lemma (3.10) below, $V_{t-r}(\lambda 1_E)(x) \geqq \pi_{t-r}(\lambda) \geqq \inf_{0<s<t} \pi_s(\lambda) > 0$, for all $x \in E$. Consequently,

$$\text{(3.8)} \qquad \overline{\lim}_{r\downarrow 0}\langle X_r, 1_E\rangle < +\infty, \qquad \text{a.s. } \mathbf{P}.$$

Given $\bar{f} \in \bar{\mathcal{R}}$, the process $1 - \exp(-e^{-\alpha t}\langle X_t, \bar{f}\rangle)$ is a bounded positive supermartingale if $\alpha > 0$ is sufficiently large. Thus $\lim_{r\downarrow 0}(1 - \exp(e^{-\alpha r}\langle X_r, \bar{f}\rangle))$ exists a.s. **P**. Because of (3.8), $\lim_{r\downarrow 0}\langle X_r, \bar{f}\rangle$ exists in $[0, +\infty[$ for all $\bar{f} \in \bar{\mathcal{R}}$. Just as in the proof of Theorem (2.11), it now follows that there is an $M(\bar{E})$-valued, $\mathcal{G}_0$-measurable random variable X_{0+} such that

$$\langle X_{0+}, \bar{f}\rangle = \lim_{r\downarrow 0} \langle X_r, \bar{f}\rangle, \qquad \forall \bar{f} \in \bar{\mathcal{R}}, \quad \text{a.s. } \mathbf{P}.$$

It is now a simple matter to check that for all $t > 0$, $F \in b\mathcal{M}(\bar{E})$, $\bar{f} \in \bar{\mathcal{R}}$,

$$\text{(3.9)} \qquad \mathbf{P}(F(X_{0+})e^{-\langle X_t, \bar{f}\rangle}) = \mathbf{P}(F(X_{0+})e^{-\langle X_{0+}, \bar{V}_t\bar{f}\rangle}),$$

which implies that $(X_{t+} : t \geqq 0)$ is Markovian under **P**, with semigroup $(\bar{Q}_t)$. We now *define* $X_0 = \int_{\bar{E}} X_{0+}(dx)\bar{P}_0(x, \cdot)$. Since $\bar{P}_0(x, \cdot)$ is carried by D for all $x \in \bar{E}$, $X_0 \in M(D)$. Since $\bar{P}_0\bar{V}_t = \bar{V}_t$, equation (3.9) is valid if X_{0+} is replaced by

X_0. Since $X_t \in M(E)$ for all $t>0$, it follows from (2.7) that $\langle X_0, 1_{D\setminus E_D}\rangle = 0$ a.s. $\mathbf{P}$. Thus the image law $N_0 := X_0(\mathbf{P})$ may be regarded as a probability measure on $(M(E_D), \mathcal{M}(E_D))$ and clearly $N_t = N_0\bar{Q}_t$.

For the uniqueness of N_0, let $\hat{\xi}$ (resp. $\hat{\phi}$) denote the restriction of ξ to E_D (resp. ϕ to $E_D \times \mathbf{R}_+$) and note ([Sh, (39.15)]) that $\hat{\xi}$ is a right process. Let $\hat{X}$ denote the ($M(E_D)$-valued) $(\hat{\xi}, \hat{\phi})$-superprocess constructed as in §2. Of course, the semigroup of $\hat{X}$ is $(\bar{Q}_t)$ restricted to $M(E_D)$. Being a right process, $\hat{X}$ satisfies the principle of uniqueness of charges ([GG, (1.1)]), which immediately implies the uniqueness of N_0. □

The use of the following lemma in the proof of Theorem (3.7) was inspired by a similar argument in §3 of [Dy4]. Theorem (3.7) itself is a variant of results in [Dy3,4]. For the statement of the lemma recall the form (1.2) of ϕ.

(3.10) LEMMA. *Let* $\gamma = \| c \|_\infty$ *and* $\hat{\beta} = \| \hat{b}^+ \|_\infty$, *where* $\hat{b} = b + \int_0^\infty un(\cdot, du)$. *Define* $\hat{\phi}(\lambda) = -\hat{\beta}\lambda - \gamma\lambda^2$ *and let* $(\pi_t(\lambda): t \geqq 0)$ *denote the unique solution of*

$$d\pi_t(\lambda)/dt = \hat{\phi}(\pi_t(\lambda)), \qquad \pi_0(\lambda) = \lambda > 0.$$

Then $\inf_{0<s<t} \pi_s(\lambda) > 0$ *and* $V_t(\lambda 1_E)(x) \geqq \pi_t(\lambda)$, *for all* λ, $t>0$ *and* $x \in E$.

PROOF. Fix λ_0, $t_0 > 0$ and put $\sigma = \lambda_0 \vee \sup_{0\leqq t\leqq t_0} \| V_t(\lambda_0 1_E) \|_\infty < \infty$. Choose $a>0$ so large that $\psi(x, \lambda) \equiv \phi(x, \lambda) + a\lambda$ is positive and increasing in $\lambda \in [0, 2\sigma]$ for each $x \in E$. Evidently $V_t(\lambda 1_E)$, $t \geqq 0$ is the unique solution of

$$V_t(\lambda 1_E)(x) = e^{-at}\lambda + \int_0^t e^{-as} P_s(x, \psi(\cdot, V_{t-s}(\lambda 1_E)))ds, \qquad t \geqq 0.$$

Now one can explicitly solve the equation defining $\pi_t(\lambda)$:

$$0 < \pi_t(\lambda) = \lambda/[(1+\gamma\lambda/\hat{\beta})e^{\hat{\beta}t} - \gamma\lambda/\hat{\beta}] < \lambda.$$

On the other hand, the solution $V_t(\lambda 1_E)$ can be obtained by Picard iteration; namely, if $v_t^1(x) := \lambda > \pi_t(\lambda)$, and

$$v_t^{n+1}(x) := e^{-at}\lambda + \int_0^t e^{-as} P_s(x, \psi(\cdot, v_{t-s}^n))ds,$$

then $v_t^n(x) \to V_t(\lambda 1_E)(x)$ as $n \to \infty$ (cf. §3 of [Dy1]). An easy induction, using the monotonicity of $\psi(x, \cdot)$, shows that if $(\lambda, t) \in [0, \lambda_0] \times [0, t_0]$, then $v_t^n(x) \geqq \pi_t(\lambda)$ for all $n \geqq 1$, and the lemma follows since $t_0 > 0$ and $\lambda_0 > 0$ were arbitrary. □

4. The infinitesimal generator L

We briefly examine certain martingales over the (ξ, ϕ)-superprocess X. For a more exhaustive study see [EK–RC], and for related results in case $\phi(x, \lambda) = \text{const.}\, \lambda^2$ see [Dy1,2].

In this section we take X to be realized on the space W of càdlàg paths from $\mathbf{R}_+$ into $M_r(E)$ that are right continuous in $M_o(E)$; otherwise the notation of previous sections is maintained.

Before discussing the infinitesimal generator L we introduce a variant of the weak infinitesimal generator, A, of ξ. Let B denote the class of finely continuous functions in $b\mathscr{E}$. Then $\mathsf{D}(A) := U^\alpha(\mathsf{B}) \subset \mathsf{B}$ is independent of $\alpha > 0$. If $g \in \mathsf{B}$ and $f = U^1 g \in \mathsf{D}(A)$, then $Af := f - g$ defines a linear operator $A : \mathsf{D}(A) \to \mathsf{B}$. Clearly $(P_t f - f)/t$ converges boundedly and pointwise to Af as $t \to 0$. Now let $\mathsf{D}(L)$ denote the class of functions on $M(E)$ of the form

$$F(\mu) = \psi(\langle \mu, f_1 \rangle, \ldots, \langle \mu, f_n \rangle),$$

where $\psi \in C_0^\infty(\mathbf{R}^n)$, $f_i \in \mathsf{D}(A)$, and $n \geqq 1$. Define L on $\mathsf{D}(L)$ by (1.3). The next result details the sense in which L is the generator of X.

(4.1) **Theorem.** *Fix $\mu \in M(E)$.*

(a) *For all $F \in \mathsf{D}(L)$,*

$$M_t^F := F(X_t) - F(X_0) - \int_0^t LF(X_s)ds, \qquad t \geqq 0, \tag{4.2}$$

is a locally bounded $\mathbf{P}_\mu$-martingale and is càdlàg a.s. $\mathbf{P}_\mu$.

(b) *Conversely, suppose $\mathbf{P}$ is a probability measure on $(W, \mathscr{G}^o)$ with $\mathbf{P}(X_0 = \mu) = 1$ such that M^F is càdlàg a.s. $\mathbf{P}$ and is a $\mathbf{P}$-local martingale for all $F \in \mathsf{D}(L)$. Then $\mathbf{P} = \mathbf{P}_\mu$.*

Proof. (a) Fix $\mu \in M(E)$ and $F \in \mathsf{D}(L)$, and note that $F(X_\cdot)$ is càdlàg a.s. $\mathbf{P}_\mu$ by virtue of (3.3) and (3.4). Clearly $LF(\nu)$ is bounded on each set $\{\nu \in M(E) : \langle \nu, 1 \rangle \leqq N\}$, so by (2.12) the integral in (4.2) is well-defined and M^F is locally bounded and càdlàg a.s. $\mathbf{P}_\mu$. A routine computation shows that

$$\lim_{t \downarrow 0} t^{-1}(Q_t(\nu, F) - F(\nu)) = L(F)(\nu),$$

the limit occurring boundedly on $\{\nu : \langle \nu, 1 \rangle \leqq N\}$ for each $N > 0$. This yields

$$Q_t(\nu, F) = Q_0(\nu, F) + \int_0^t Q_s(\nu, LF)ds, \qquad t \geqq 0,$$

which implies that M^F is a $\mathbf{P}_\mu$-martingale.

(b) Let $\mathsf{D}_0(A) = \bigcup_{\alpha>0} U^\alpha(\mathsf{D}(A))$; if $f \in \mathsf{D}_0(A)$ then $t \mapsto P_t f(x)$ is continuously differentiable for each $x \in E$. A glance at (2.2) reveals that the same is true of $t \mapsto V_t f(x)$. Arguing now as in the proof of [EK–RC, II.1.1], we see that if each M^F, $F \in \mathsf{D}(L)$, is a $\mathbf{P}$-local martingale (càdlàg a.s. $\mathbf{P}$), then so is each process $(\exp(-\langle X_t, V_{s-t}f\rangle) : t \in [0, s])$, $f \in p\mathsf{D}_0(A)$. Since $\mathsf{D}_0(A)$ is dense in $\mathsf{D}(A)$ (which in turn is dense in B) for the topology of bounded pointwise convergence, the conclusion of the last sentence is valid for all $f \in p\mathsf{B}$. This implies that (X_t) under $\mathbf{P}$ is Markovian with semigroup (Q_t) (see [EK–RC, Thm. II.1.5] or [Sh, (7.4)]), hence $\mathbf{P} = \mathbf{P}_\mu$. □

(4.3) COROLLARY. *Fix $\mu \in M(E)$. Each locally square integrable $\mathbf{P}_\mu$-local martingale M has predictable quadratic variation $\langle M, M\rangle$ which is absolutely continuous with respect to Lebesgue measure. In particular, if $F, G \in \mathsf{D}(L)$ then*

$$\langle M^F, M^G\rangle_t = \int_0^t \Gamma(F, G)(X_s)ds, \tag{4.4}$$

where the "gradient squared" operator Γ is defined on $\mathsf{D}(L) \times \mathsf{D}(L)$ by

$$\Gamma(F, G) = L(FG) - FL(G) - GL(F). \tag{4.5}$$

PROOF. Since $\mathsf{D}(L)$ is an algebra and each M^F is locally square integrable, it is standard that $\langle M^F, M^G\rangle$ exists and takes the form (4.4); see [DM2, XV.22]. Using (4.1b), a stable subspace argument (see [DM2, XV.26]) now shows that each locally square integrable $\mathbf{P}_\mu$-local martingale has absolutely continuous quadratic variation. □

The Lévy system of X has been computed in [EK–RC] and used there to characterize those ϕ's for which X has continuous paths. We prove an analogous result, making use of a lemma of Bakry and Emery. This lemma is valid in considerable generality but we state it now only for the process X. The notation is that of (4.1) and (4.3).

(4.6) LEMMA ([BE, Prop. 2]). *Suppose that Γ has the derivation property:*

$$\Gamma(FG, H) = F\Gamma(G, H) + G\Gamma(F, H), \qquad \forall F, G, H \in \mathsf{D}(L).$$

Then the processes M^F and $F(X_\cdot)$ have continuous paths a.s. $\mathbf{P}_\mu$, for all $F \in \mathsf{D}(L)$ and all $\mu \in M(E)$.

(4.7) COROLLARY. *Let $N = \{x \in E : n(x, \cdot) \neq 0\}$.*

(a) *If* $U^1 1_N = 0$, *then* $t \mapsto X_t$ *is continuous in* $M_r(E)$ (*and continuous in* $M_o(E)$ *if* ξ *is a Hunt process*) *a.s.* $\mathbf{P}_\mu$ *for all* $\mu \in M(E)$.

(b) *Conversely, if* $t \mapsto \langle X_t, 1 \rangle$ *is continuous a.s., then* $U^1 1_N = 0$.

PROOF. (a) Because of (2.7), $U^1 1_N = 0$ if and only if $\mathbf{P}_\mu(\int_0^\infty \langle X_t, 1_N \rangle dt) = 0$, for all $\mu \in M(E)$. Suppose that $U^1 1_N = 0$. We can redefine L, dropping the third term on the L.H.S. of (1.3), without affecting the validity of (4.1), (4.3), or (4.6) — see Remark (2.5b). This change being made, we have

$$\Gamma(F, G)(\mu) = 2 \int_E \mu(dx) c(x) F'(\mu; x) G'(\mu; x),$$

where $F'(\mu; x)$ is as in (1.3) and G' is defined analogously. Clearly Γ has the derivation property, so by (4.6), $t \mapsto \langle X_t, f \rangle$ is continous a.s. for all $f \in \mathrm{D}(A)$. But any $f \in \mathrm{B}$ is the bounded pointwise limit of a sequence from $\mathrm{D}(A)$. It follows that $\langle X_\cdot, f \rangle$ is predictable for all $f \in \mathrm{B}$, hence for all $f \in b\mathscr{E}$ by the monotone class theorem. But for such f the predictable projection of $\langle X_\cdot, f \rangle$ is $\langle X^r_-, f \rangle$ because of (2.18). Thus $X^r_- = X$ a.s. Moreover, if ξ is a Hunt process, then by (3.6) the same reasoning yields $X^o_- = X$, where X^o_- is the left limit of X in $M_o(E)$.

(b) Assume that $\langle X_\cdot, 1 \rangle$ is continuous a.s. Fix $\mu \in M(E)$. By (4.1a),

$$\exp(-\langle X_t, 1 \rangle) - \exp(-\langle X_0, 1 \rangle) - \int_0^t L(e_1)(X_s) ds$$

is a $\mathbf{P}_\mu$-martingale. On the other hand, by (4.1a), (4.3), and Itô's formula,

$$\exp(-\langle X_t, 1 \rangle) - \exp(-\langle X_0, 1 \rangle) - \int_0^t \langle X_s, b + \hat{c}/2 \rangle ds$$

is a $\mathbf{P}_\mu$-martingale. Subtracting we see that the *decreasing* process

$$I_t = \int_0^t ds \int_E X_s(dx) \int_0^\infty (1 - e^{-u} - u) n(x, du)$$

is a $\mathbf{P}_\mu$-martingale with $I_0 = 0$. Thus I is $\mathbf{P}_\mu$-evanescent and since $1 - e^{-u} - u < 0$ if $u > 0$,

$$\mathbf{P}_\mu\left(\int_0^\infty \langle X_t, 1_N \rangle dt\right) = 0.$$

Consequently $U^1 1_N = 0$, by the remark at the beginning of the proof. □

REMARKS. (a) The method used to prove (4.7b) is adapted from [EK–RC].

(b) Under the condition $U^1 1_N = 0$ of (4.7a), every càdlàg $\mathbf{P}_\mu$-local martingale has continuous paths a.s. $\mathbf{P}_\mu$ for all $\mu \in M(E)$ (cf. [D1, Th. 7.2]).

5. Complements

In this final section we discuss how certain of our hypotheses can be weakened.

First, the hypothesis $P_t 1 = 1$ can be dropped. If (P_t) is only sub-Markovian then we introduce a cemetery state ∂ as usual and extend (P_t) to a Markov semigroup (P_t^∂) on $E \cup \{\partial\}$. Let X^∂ be the (ϕ, ξ^∂)-superprocess, where $\phi(\partial, \cdot) = 0$ and ξ^∂ is the right process on $E \cup \{\partial\}$ with semigroup (P_t^∂). Since ∂ is a trap for ξ^∂, the process $X := X^\partial|_E$ is a right Markov process with state space $M_r(E)$ (or $M_o(E)$) and semigroup (Q_t) satisfying (2.1), $V_t f$ being determined by (P_t) as before. (The right continuity of X in $M_r(E)$ (or $M_o(E)$) follows from the fact that 1_E is (P_t^∂)-excessive.) However, the results of §2–4 concerning X_{t-}^r may fail without further hypotheses. We leave the consideration of this point to the interested reader.

A second extension involves dropping the condition $\phi(x, 0) = 0$, which is implicit in (1.2). Suppose that $\tilde{\phi}(x, \lambda) = \phi(x, \lambda) + a(x)$ where ϕ is as in previous sections and $a \in bp\mathscr{E}$. Let X be the (ξ, ϕ)-superprocess on $M(E)$. Now kill X according to the multiplicative functional

$$K_t = \exp\left(-\int_0^t \langle X_s, a\rangle ds\right).$$

The resulting subprocess $\tilde{X}$ has state space $M(E) \cup \{\Delta\}$, where Δ is the cemetery for $\tilde{X}$. The sub-Markovian semigroup $(\tilde{Q}_t)$ of $\tilde{X}$ is determined on $M(E)$ by

$$\tilde{Q}_t(\mu, e_f) = \mathbf{P}_\mu(K_t e_f(X_t)). \tag{5.1}$$

Using (2.3) it is not hard to check that

$$\tilde{Q}_t(\mu, e_f) = \exp(-\langle \mu, W_t(f, a)\rangle). \tag{5.2}$$

But then $\tilde{V}_t f := W_t(f, a)$ satisfies

$$\tilde{V}_t f(x) = P_t f(x) + \int_0^t P_s(x, \tilde{\phi}(\cdot, \tilde{V}_{t-s} f))ds, \qquad t \geqq 0,$$

so we are justified in calling $\tilde{X}$ a $(\xi, \tilde{\phi})$-superprocess. Of course $\tilde{X}$ has lifetime $\tilde{\zeta}$ (say), but owing to the continuity of (K_t), $\tilde{\zeta}$ is totally inaccessible. Thus, just as in previous sections, X is a right process, a Hunt process when viewed as having state space $M_r(E) \cup \{\Delta\}$, and an $M_o(E) \cup \{\Delta\}$-valued Hunt process if ξ is a Hunt process. Also, the results of §4 are valid once the obvious modifications have been made to account for the finiteness of the lifetime $\tilde{\zeta}$; cf.

[EK–RC, §II]. Alternatively, formulas (5.1) and (5.2) may be viewed as characterizing the joint law of X_t and the "weighted occupation time" $Y_t := \int_0^t X_s\,ds$. Namely,

$$\mathbf{P}_\mu(\exp(-\langle X_t, f\rangle - \langle Y_t, g\rangle)) = \exp(-\langle \mu, W_t(f, g)\rangle).$$

Thus the measure $\Pi_t(\mu;\cdot)$ of Corollary (2.6) is just the $\mathbf{P}_\mu$-law of (X_t, Y_t). See Iscoe [I1,2] for a study of (Y_t) in case ξ is a symmetric stable process in $\mathbf{R}^d$ and $\phi(x, \lambda) = -\text{const.}\,\lambda^\alpha$, $1 < \alpha \leqq 2$.

Finally, at times one would like to give X an initial state X_0 in the class of σ-finite measures of infinite total mass (cf. [Dy3,4; EK–RC; I1,2]). Formula (1.1) makes it clear how this can be accomplished: given such a (nonrandom) choice of $\mu = X_0$, decompose μ as a sum $\mu = \Sigma\,\mu_n$ of finite measures, let $\{X^{(n)} : n \geqq 1\}$ be independent (ξ, ϕ)-superprocesses with $X_0^{(n)} = \mu_n$, and put $X = \Sigma\,X^{(n)}$. It is intuitively obvious (and not hard to check) that X is a strong Markov process with quasi-left continuous natural filtration. Moreover, Lemma (3.10) implies that if $\langle X_0, 1\rangle = \infty$ then $\langle X_t, 1\rangle = \infty$ for all $t > 0$ almost surely. However, without "Feller" hypotheses on the semigroup of ξ, the path regularity of X becomes a more delicate issue in this setting, as does the "martingale problem" characterization of §4.

Appendix

Let Π be a probability measure on $(M(E), \mathcal{M}(E))$, where $(E, \mathcal{E})$ is a Lusin space. The *Laplace functional* L_Π of Π is defined by

$$L_\Pi(f) = \int_{M(E)} \Pi(d\mu)e^{-\langle \mu, f\rangle}, \qquad f \in bp\mathcal{E}.$$

By monotone convergence, if $(f_n) \subset bp\mathcal{E}$ with $f_n \downarrow f$, then $L_\Pi(f_n) \uparrow L_\Pi(f)$. Moreover, it is easy to check that L_Π is positive definite (see (A.1) below). For a detailed discussion of Laplace functionals, see Kallenberg [K].

In this appendix we shall show that the necessary conditions noted above are also sufficient for a function $\varphi : S \to [0, 1]$ to be the Laplace functional of a subprobability measure on $(M(E), \mathcal{M}(E))$. Actually we prove a more general result which admits (2.6) as an immediate corollary. Undoubtedly this characterization is well-known, but we know of no reference in the literature. It will be deduced from a general integral representation theorem for positive definite functions on semigroups; for this material, the monograph of Berg, Christensen, and Ressel [BCR] is an excellent source.

Let $(S, +)$ be an Abelian semigroup with neutral element 0. We assume that S is 2-divisible: each $t \in S$ can be written as $t = s + s$ for some $s \in S$. (The semigroup we have in mind is the product $bp\mathscr{E} \times bp\mathscr{E}$, where $bp\mathscr{E}$ is viewed as a semigroup under pointwise addition.)

(A.1) DEFINITION. A function $\varphi : S \to \mathbf{R}$ is *positive definite* if

$$\sum_{i,j} a_i a_j \varphi(s_i + s_j) \geqq 0 \tag{A.2}$$

for all $\{a_1, \ldots, a_n\} \subset \mathbf{R}$, $\{s_1, \ldots, s_n\} \subset S$, and $n \geqq 1$. If (A.2) holds for $\varphi = -\psi$ whenever the a_i satisfy the side condition $\Sigma_i a_i = 0$, then ψ is *negative definite*.

As noted in §2 the function ϕ defined by (1.2) is negative definite.

(A.3) PROPOSITION ([BCR, p.133]). *Let $\psi : S \to \mathbf{R}$. Then ψ is negative definite if and only if* $\exp(-t\psi)$ *is positive definite for all $t > 0$.*

A *bounded semicharacter* of S is a function $\rho : S \to [-1, 1]$ such that $\rho(0) = 1$, and $\rho(s + t) = \rho(s)\rho(t)$ for all $s, t \in S$. (Since S is 2-divisible, $\rho(s) \geqq 0$, $\forall s \in S$ for all such ρ.) The class $\hat{S}$ of all bounded semicharacters is an Abelian semigroup under pointwise multiplication with neutral element the constant semicharacter 1. Clearly S is a compact Hausdorff topological semigroup under the topology of pointwise convergence.

(A.4) THEOREM ([BCR, p.96]). *Let $\varphi : S \to \mathbf{R}$ be a bounded positive definite function with $\varphi(0) \geqq 0$. Then there is a unique positive Radon measure γ on $\hat{S}$ such that*

$$\varphi(s) = \int_{\hat{S}} \rho(s)\gamma(d\rho), \qquad \forall s \in S.$$

(A.5) COROLLARY ([BCR, p.110]). *Let $\varphi : S \to \mathbf{R}$ be negative definite and lower bounded. Then there exist an additive function $\alpha : S \to \mathbf{R}_+$ and a positive Radon measure κ on $\hat{S} \setminus \{1\}$ such that*

$$\varphi(s) = \varphi(0) + \alpha(s) + \int_{\hat{S}\setminus\{1\}} (1 - \rho(s))\kappa(d\rho), \qquad \forall s \in S.$$

The folowing result contains Corollary (2.6) as a special case.

(A.6) COROLLARY. *Let $\varphi : bp\mathscr{E} \times bp\mathscr{E} \to [0, 1]$ be positive definite and such that $\varphi(f_n, g_n) \uparrow \varphi(f, g)$ whenever $(f_n), (g_n) \subset bp\mathscr{E}$ and $f_n \downarrow 0$, $g_n \downarrow 0$ pointwise. Then there is a unique subprobability measure Π on $(M(E), \mathscr{M}(E))^2$ such that*

$$\int \Pi(d\mu, d\nu) e_f(\mu) e_g(\nu) = \varphi(f, g), \qquad \forall f, g \in bp\mathscr{E}.$$

PROOF. We apply (A.4) with $S = bp\mathscr{E} \times bp\mathscr{E}$. Let γ represent φ as in the theorem and put

$$\hat{S}_+ = \{\rho \in \hat{S} : \rho(1, 1) > 0\} = \{\rho \in \hat{S} : \rho(1, 0)\rho(0, 1) > 0\}.$$

Clearly $\hat{S}_+$ is a Borel set and $\rho \in \hat{S}_+$ iff $\rho(f, g) > 0$ for all $f, g \in bp\mathscr{E}$. Moreover

$$\begin{aligned} \gamma(\hat{S}) &= \varphi(0, 0) = \lim_{n \to \infty} \varphi(0, 1/n) \\ &= \lim_{n \to \infty} \int_{\hat{S}} \rho(0, 1/n) \gamma(d\rho) = \lim_{n \to \infty} \int_{\hat{S}} \rho(0, 1)^{1/n} \gamma(d\rho) \\ &= \gamma(\{\rho \in \hat{S} : \rho(0, 1) > 0\}) \end{aligned}$$

and similarly $\gamma(\hat{S}) = \gamma(\{\rho \in \hat{S} : \rho(1, 0) > 0\})$. Thus $\gamma(\hat{S} \setminus \hat{S}_+) = 0$. If $\rho \in \hat{S}_+$, then we define positive additive functionals on $bp\mathscr{E}$ by

$$k_1(\rho, f) = -\log \rho(f, 0), \qquad k_2(\rho, f) = -\log \rho(0, f).$$

Note that $k_i(\rho, \cdot)$ is homogeneous over the positive rationals. Also, the sequential continuity hypothesis imposed on φ implies that if $f_n \downarrow 0$ then $k_i(\rho, f_n) \downarrow 0$ for γ-a.e. $\rho \in \hat{S}_+$. A standard result ([G2]) on the regularization of pseudo-kernels now yields the existence of Borel measurable mappings $\bar{k}_i : \hat{S}_+ \to M(E)$, $i = 1, 2$, such that

$$\gamma\{\rho \in \hat{S}_+ : \bar{k}_i(\rho, f) \neq k_i(\rho, f)\} = 0, \qquad \forall f \in bp\mathscr{E}.$$

The image of $\gamma|_{\hat{S}_+}$ under the mapping $\rho \mapsto (\bar{k}_1(\rho, \cdot), \bar{k}_2(\rho, \cdot))$ is the desired subprobability measure Π. For the uniqueness of Π see [K, Th.3.1]. □

Corollary (2.6) now follows upon taking $\varphi(f, g) = \exp(-\langle \mu, W_t(f, g)\rangle)$ in (A.6), using (2.3e) and (A.3).

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